

SELF-INTERLACING POLYNOMIALS

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ABSTRACT. We describe a new subclass of the class of real polynomials with real simple roots called self-interlacing polynomials. This subclass is isomorphic to the class of real Hurwitz stable polynomials (all roots in the open left half-plane). In the work, we present basic properties of self-interlacing polynomials and their relations with Hurwitz and Hankel matrices as well as with Stiltjes type of continued fractions. We also establish “self-interlacing” analogues of the well-known Hurwitz and Liénard-Chipart criterions for stable polynomials. A criterion of Hurwitz stability of polynomials in terms of minors of certain Hankel matrices is established.

1. INTRODUCTION

A real polynomial $p(z)$ is called *self-interlacing* if all its roots are real and simple (of multiplicity one) and interlace the roots of the polynomial $p(-z)$.

In other word, if λ_j , $j = 1, \dots, n$, are the roots of the polynomial $p(z)$ ordered as decreasing their absolute values, then they satisfy one of the following inequalities

$$(1.1) \quad \lambda_1 > -\lambda_2 > \lambda_3 > \dots > (-1)^{n-1} \lambda_n > 0,$$

or

$$(1.2) \quad -\lambda_1 > \lambda_2 > -\lambda_3 > \dots > (-1)^n \lambda_n > 0.$$

If the roots of the polynomial $p(z)$ are distributed as in (1.1), then $p(z)$ is called the self-interlacing polynomial of kind I . Respectively, if the roots of $p(z)$ are distributed as in (1.2), then $p(z)$ is called self-interlacing polynomial of kind II . We denote the class of all self-interlacing polynomials as **SI**. Respectively, the classes of all self-interlacing polynomials of kind I and II are denoted as **SI_I** and **SI_{II}**, respectively.

Implicitly, the self-interlacing polynomials seem to appear first time in [4, Lemma 2.6] (in its first edition) where a necessary condition for real polynomials to be self-interlacing was established (see Theorem 3.4 of the present work). In [9] (see also [18] and references there) there were introduced tridiagonal matrices of the form:

$$(1.3) \quad \begin{pmatrix} b_1 & b_2 & 0 & \dots & 0 & 0 \\ b_2 & 0 & b_3 & \dots & 0 & 0 \\ 0 & b_3 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & b_n \\ 0 & 0 & 0 & \dots & b_n & 0 \end{pmatrix}, \quad b_1 \neq 0, \quad b_k > 0, \quad k = 2, \dots, n,$$

whose characteristic polynomials belong to **SI_I** if $b_1 > 0$ (it was established implicitly without mentioning of self-interlacing polynomials). In [9], the author also proved that for any polynomial $p \in \mathbf{SI}_I$, there exists a unique matrix of the form (1.3) with $b_1 > 0$ whose characteristic polynomial is p . Clearly, for $b_1 < 0$, we deal with the polynomials in the class **SI_{II}**.

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It is easy to see that $p(z) \in \mathbf{SI}_I$ if, and only if, $p(-z) \in \mathbf{SI}_{II}$, so it is sufficient to study only one of the classes, say, \mathbf{SI}_I .

One of the main results of the present work is the following theorem.

Theorem 1.1. *The class \mathbf{SI}_I is isomorphic to the class of real Hurwitz stable polynomials.*

Recall that a real polynomial is called Hurwitz stable (or stable) if its zeroes lie in the *open* left half-plane of the complex plane. The mentioned isomorphism is a linear operator acting on the coefficients of polynomials. Precisely, a polynomial

$$(1.4) \quad p(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + a_3 z^{n-3} + a_4 z^{n-4} + \dots + a_n$$

belongs to \mathbf{SI}_I if, and only if, the polynomial

$$(1.5) \quad q(z) = a_0 z^n - a_1 z^{n-1} - a_2 z^{n-2} + a_3 z^{n-3} + a_4 z^{n-4} - \dots + (-1)^{\frac{n(n+1)}{2}} a_n$$

is Hurwitz stable (see Theorem 3.1).

Due to the isomorphism self-interlacing polynomials have some properties similar to ones of Hurwitz stable polynomials. In this work, we prove the following analogue of Hurwitz criterion.

Theorem 1.2 (analogue of Hurwitz's criterion). *A real polynomial p belongs to the class \mathbf{SI}_I if, and only if, its Hurwitz minors $\Delta_k(p)$ satisfy the inequalities*

$$(-1)^j \Delta_{2j-1}(p) > 0, \quad j = 1, \dots, \left\lfloor \frac{n+1}{2} \right\rfloor,$$

$$\Delta_{2j}(p) > 0, \quad j = 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor,$$

where $\lfloor \rho \rfloor$ denotes the largest integer not exceeding ρ .

Recall that for a real polynomial p defined in (1.4), its *Hurwitz minors* $\Delta_k(p)$ have the form

$$(1.6) \quad \Delta_k(p) = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & \dots & a_{2k-1} \\ a_0 & a_2 & a_4 & a_6 & \dots & a_{2k-2} \\ 0 & a_1 & a_3 & a_5 & \dots & a_{2k-3} \\ 0 & a_0 & a_2 & a_4 & \dots & a_{2k-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_k \end{vmatrix}, \quad k = 1, \dots, n,$$

where we set $a_i \equiv 0$ for $i > n$.

Another criterion for stability of real polynomials whose analogue also can be established for self-interlacing polynomials is the Liénard–Chipart criterion [14] (see also [6, 8]).

Theorem 1.3 (analogue of Liénard–Chipart's criterion). *A real polynomial p belongs to the class \mathbf{SI}_I if, and only if, the following inequalities hold*

$$(-1)^j \Delta_{2j-1}(p) > 0, \quad j = 1, \dots, \left\lfloor \frac{n+1}{2} \right\rfloor,$$

$$(-1)^j a_{2j} > 0, \quad j = 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor.$$

As well, we provide one more criterion for polynomials to be self-interlacing (Theorem 2.1) which together with Theorem 1.1 and formulæ (1.4)–(1.5) provides a seemingly new criterion of stability of real polynomials. Namely, a polynomial $p(z)$ of degree n is Hurwitz stable if, and only if, the following inequalities hold (Theorem 5.3)

$$(-1)^{\frac{j(j+1)}{2}} D_j(R) > 0, \quad j = 1, \dots, n,$$

where $D_j(R)$ are the Hankel minors

$$D_j(R) = \begin{vmatrix} s_0 & s_1 & s_2 & \cdots & s_{j-1} \\ s_1 & s_2 & s_3 & \cdots & s_j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{j-1} & s_j & s_{j+1} & \cdots & s_{2j-2} \end{vmatrix}, \quad j = 1, 2, 3, \dots,$$

constructed with the coefficient of the Laurent series of the function

$$R(z) := \frac{(-1)^n p(-z)}{p(z)} = 1 + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \dots,$$

It is interesting to note that the function $R(z)$ related to a Hurwitz stable polynomial maps the open right half-plane of the complex plane to the open unit disc (Theorem 5.3), while $R(z)$ related to a self-interlacing polynomial of kind I maps the upper half-plane to the lower half-plane (Theorem 2.1).

Self-interlacing polynomials appear in the theory of orthogonal polynomials. For example, the Chebyshev polynomials of III and IV kinds are self-interlacing, see e.g. [15] for roots distribution of these polynomials. Self-interlacing polynomials also appear as characteristic polynomials of certain structured matrices. We postpone the study of matrices with self-interlacing spectra and corresponding orthogonal polynomials to other parts of the work.

Finally, we would like to inform the reader that instead of citing various papers regarding rational functions, matrices and especially rational R -functions, we cite the survey [8] that was written exactly for such quotations. All other references can be found there. Some results of this work were mentioned in the technical report [19], but here we substantially simplified most of the proofs and provided additional properties of self-interlacing polynomials.

The paper is organized as follows. In Section 2 we introduce the basic objects and prove basic theorems. Section 3 is devoted to Theorem 1.1 and its consequences. In Section 4 we study some properties of self-interlacing polynomials. In particular, we study the minors of Hurwitz matrix of self-interlacing polynomials. In Section 5, we establish some curious determinant formulæ and prove the stability criterion mentioned above.

2. HURWITZ AND LIÉNARD–CHIPART CRITERIONS

Consider a real polynomial

$$(2.1) \quad p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \quad a_1, \dots, a_n \in \mathbb{R}, \quad a_0 > 0.$$

In the rest of the paper, we use the following notation

$$(2.2) \quad l \stackrel{\text{def}}{=} \left\lfloor \frac{n+1}{2} \right\rfloor,$$

where $n = \deg p$, and $[\rho]$ denotes the largest integer not exceeding ρ .

We introduce the following two auxiliary rational functions associated with the polynomial $p(z)$:

$$(2.3) \quad z\Phi(z^2) = \frac{p(z) - (-1)^n p(-z)}{p(z) + (-1)^n p(-z)} = \frac{a_1 z^{n-1} + a_3 z^{n-3} + a_5 z^{n-5} + \cdots}{a_0 z^n + a_2 z^{n-2} + a_4 z^{n-4} + \cdots},$$

and

$$(2.4) \quad R(z) = \frac{(-1)^n p(-z)}{p(z)}.$$

Note that for odd n the function $\Phi(u)$ has a pole at zero.

It is easy to see that these functions are related as follows

$$(2.5) \quad z\Phi(z^2) = \frac{1 - R(z)}{1 + R(z)},$$

and

$$(2.6) \quad R(z) = \frac{1 - z\Phi(z^2)}{1 + z\Phi(z^2)}.$$

It turns out that if the polynomial $p(z)$ is self-interlacing, the functions $\Phi(u)$ and $R(z)$ possess some remarkable properties. Namely, the following two theorems hold.

Theorem 2.1. *Let $p(z)$ be a real polynomial. $p(z) \in \mathbf{SI}_I$ if, and only if, the function $R(z)$ defined in (2.4) maps the upper half-plane of the complex plane to the lower half-plane and has exactly n poles.*

Proof. Let the polynomial $p(z)$ is self-interlacing of kind I . Then by definition, the roots and poles of the function $R(z)$ are real, simple and interlacing. In particular, $R(z)$ has exactly n poles. Thus $R(z)$ maps the upper half-plane to itself or to the lower half-plane, and it is monotone on the real line between its poles (see e.g. [8, Chapter 3]). But $R(z) \rightarrow 1$ as $z \rightarrow +\infty$, and its largest zero $-\lambda_2(> 0)$ is smaller than its largest pole $\lambda_1(> 0)$, where λ_j are the roots of $p(z)$, so $R(z) > 1$ on the interval $(\lambda_1, +\infty)$. Consequently, $R(z)$ is decreasing on the real line and maps the upper half-plane to the lower half-plane (see [8]), as required.

Conversely, if $R(z)$ maps the upper half-plane to the lower half-plane and has exactly n poles, then the roots of its numerator $p(-z)$ and denominator $p(z)$ are real, simple and interlacing (see [8, Theorem 3.4]), and $p(z)$ and $p(-z)$ have no common roots. Moreover, since $R(z)$ is decreasing on the real line, its largest pole (the largest root of $p(z)$) is greater than its largest root (the largest root of $p(-z)$), therefore, $p(z) \in \mathbf{SI}_I$. \square

The function $R(z)$ has interesting properties even for arbitrary polynomial $p(z)$ but for self-interlacing polynomials this function plays the most important role.

The following theorem will give us a tool to prove an analogue of the famous Hurwitz stability criterion for self-interlacing polynomials.

Theorem 2.2. *Let p be a real polynomial of even (odd) degree n as in (2.1). The polynomial $p \in \mathbf{SI}_I$ is self-interlacing if, and only if, its associated function Φ defined in (2.3) maps the upper half-plane to itself and has only positive (nonnegative) poles.*

Proof. Indeed, let $p \in \mathbf{SI}_I$, so $p(z)$ and $p(-z)$ have no common roots. Then by Theorem 2.1 the function $R(z)$ defined in (2.4) maps the upper half-plane to the lower half-plane. At the same time, the function $\frac{1-w}{1+w}$ maps the lower half-plane to the upper half-plane as it is easy to check. Thus, from (2.5) we obtain that the function $z\Phi(z^2)$ maps the upper half-plane to itself. Therefore (see e.g. [8, Theorem 3.4]), $z\Phi(z^2)$ has the form

$$(2.7) \quad z\Phi(z^2) = \sum_{k=1}^r \frac{\alpha_k}{\mu_k - z} - \sum_{k=1}^r \frac{\alpha_k}{\mu_k + z} - \frac{\alpha_0}{z},$$

where $\alpha_0 \geq 0$ ($= 0$ if, and only if $n = 2r$), $\alpha_k > 0$, $k = 1, \dots, r$,

$$(2.8) \quad r \stackrel{\text{def}}{=} \left\lceil \frac{n}{2} \right\rceil,$$

and

$$0 < \mu_1 < \mu_2 < \dots < \mu_r.$$

Here we take into account the facts that $z\Phi(z^2)$ is an odd function, and that the degree of the numerator of $z\Phi(z^2)$ is $n = \deg p$, while the degree of its denominator is $n - 1$. Thus the function $\Phi(u)$ has the form

$$(2.9) \quad \Phi(u) = -\frac{\beta_0}{u} + \sum_{k=1}^r \frac{\beta_k}{\omega_k - u},$$

where $\beta_0 = \alpha_0 \geq 0$ ($= 0$ if, and only if $n = 2r$), $\beta_k = 2\alpha_k > 0$, and $\omega_k = \mu_k^2 > 0$, $k = 1, \dots, r$. Here r is defined in (2.8). So (see [8, Theorem 3.4]) the function $\Phi(u)$ maps the upper half-plane to itself, and all its poles are simple and positive for $n = 2r$ or nonnegative for $n = 2r + 1$, as required.

Conversely, if $\Phi(u)$ maps the upper half-plane to itself, and has only positive poles whenever $n = 2r$ and nonnegative poles whenever $n = 2r + 1$, then by Theorem 3.4 from [8], the function $\Phi(u)$ has the form (2.9) with positive β_k , distinct positive ω_k , $k = 1, \dots, r$, and nonnegative β_0 (which is zero only for even n). Here we took into account that the degree of its numerator is less than the degree of its denominator as it follows from (2.3). Thus, $z\Phi(z^2)$ can be presented as in (2.7), where $\lambda_k = \sqrt{\omega_k} > 0$, and $\alpha_k = \frac{\beta_k}{2} > 0$, $k = 1, \dots, r$, $\alpha_0 = \beta_0 \geq 0$ ($= 0$ if, and only if, $n = 2r$). Therefore [8, Chapter 3], the function $z\Phi(z^2)$ maps the upper half-plane to itself, so by (2.6) the function $R(z)$ maps the upper half-plane to the lower half-plane and has exactly n poles. Now Theorem 2.1 implies $p \in \mathbf{SI}_l$. \square

Remark 2.3. Another, more complicated, proof of Theorem 2.2 can be found in the technical report [19] (see Theorem 4.3 there).

Note that the degrees of the numerator and denominator of $\Phi(u)$ are, respectively, $l - 1$ and l , where l is defined in (2.2), so $\Phi(u)$ tends to zero as u tends to infinity. Expand now the function $\Phi(u)$ into Laurent series at infinity:

$$(2.10) \quad \Phi(u) = \frac{p_1(u)}{p_0(u)} = \frac{s_0}{u} + \frac{s_1}{u^2} + \frac{s_2}{u^3} + \frac{s_3}{u^4} + \dots,$$

and construct two sequences of Hankel determinants:

$$(2.11) \quad D_j(\Phi) = \begin{vmatrix} s_0 & s_1 & s_2 & \dots & s_{j-1} \\ s_1 & s_2 & s_3 & \dots & s_j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{j-1} & s_j & s_{j+1} & \dots & s_{2j-2} \end{vmatrix}, \quad j = 1, 2, 3, \dots,$$

and

$$(2.12) \quad \widehat{D}_j(\Phi) = \begin{vmatrix} s_1 & s_2 & s_3 & \dots & s_j \\ s_2 & s_3 & s_4 & \dots & s_{j+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_j & s_{j+1} & s_{j+2} & \dots & s_{2j-1} \end{vmatrix}, \quad j = 1, 2, 3, \dots,$$

using the coefficients of the expansion (2.10).

From the results of [8, Chapter 3] (see references there) one can obtain the following lemmata.

Lemma 2.4. *The function $\Phi(u)$ maps the upper half-plane of the complex plane to itself and has exactly l poles if, and only if, the following inequalities hold*

$$(-1)^j D_j(\Phi) > 0, \quad j = 1, \dots, l,$$

$$D_j(\Phi) = 0, \quad j > l.$$

Lemma 2.5. *If the function $\Phi(u)$ maps the upper half-plane of the complex plane to itself, then it has only positive poles if, and only if, the following inequalities hold*

$$(2.13) \quad (-1)^j \widehat{D}_j(\Phi) > 0, \quad j = 1, \dots, l.$$

$$\widehat{D}_j(\Phi) = 0, \quad j > l.$$

where l is the number of poles of the function $\Phi(u)$.

Lemma 2.6. *If the function $\Phi(u)$ maps the upper half-plane of the complex plane to itself, then it has only nonnegative poles if, and only if, the following inequalities hold*

$$(2.14) \quad (-1)^j \widehat{D}_j(\Phi) > 0, \quad j = 1, \dots, l-1.$$

$$\widehat{D}_j(\Phi) = 0, \quad j \geq l.$$

where l is the number of poles of the function $\Phi(u)$.

Applying the famous Hurwitz formula (see [6, p. 214], [8, Theorem 1.5] and references there) for determinants to the function $\Phi(u)$, we get the following

$$(2.15) \quad D_j(\Phi) = \frac{1}{a_0^{2j}} \begin{vmatrix} a_0 & a_2 & a_4 & a_6 & a_8 & \dots & a_{4j-2} \\ 0 & a_1 & a_3 & a_5 & a_7 & \dots & a_{4j-3} \\ 0 & a_0 & a_2 & a_4 & a_6 & \dots & a_{4j-4} \\ 0 & 0 & a_1 & a_3 & a_5 & \dots & a_{4j-5} \\ 0 & 0 & a_0 & a_2 & a_4 & \dots & a_{4j-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & a_{2j-1} \end{vmatrix} = \frac{1}{a_0^{2j-1}} \Delta_{2j-1}(p), \quad j = 1, \dots, l,$$

where l defined in (2.2), and $\Delta_j(p)$ are defined in (1.6). Furthermore, noting that $\widehat{D}_j(\Phi) = D_j(u\Phi(u))$, $j = 1, 2, \dots$ we obtain

$$(2.16) \quad \widehat{D}_j(\Phi) = \frac{1}{a_0^{2j}} \begin{vmatrix} a_0 & a_2 & a_4 & a_6 & \dots & a_{4j-4} & a_{4j-2} \\ a_1 & a_3 & a_5 & a_7 & \dots & a_{4j-3} & a_{4j-1} \\ 0 & a_0 & a_2 & a_4 & \dots & a_{4j-6} & a_{4j-4} \\ 0 & a_1 & a_3 & a_5 & \dots & a_{4j-5} & a_{4j-3} \\ 0 & 0 & a_0 & a_2 & \dots & a_{4j-8} & a_{4j-6} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{2j-2} & a_{2j} \\ 0 & 0 & 0 & 0 & \dots & a_{2j-1} & a_{2j+1} \end{vmatrix} = \frac{(-1)^j}{a_0^{2j}} \Delta_{2j}(p), \quad j = 1, \dots, r,$$

where r defined in (2.8).

Thus, from Theorem 2.2, Lemmata 2.4–2.6 and formulæ (2.15)–(2.16), we obtain the following criterion of self-interlacing, which is an analogue of the Hurwitz stability criterion.

Theorem 1.2. *A real polynomial p of degree n as in (2.1) belongs to the class \mathbf{SI}_I if, and only if, the Hurwitz minors $\Delta_j(p)$ satisfy the inequalities:*

$$(2.17) \quad (-1)^j \Delta_{2j-1}(p) > 0, \quad j = 1, \dots, l,$$

$$(2.18) \quad \Delta_{2j}(p) > 0, \quad j = 1, \dots, r,$$

where l and r are defined in (2.2) and (2.8), respectively.

Proof. Let $n = 2l$. Then by Theorem 2.2, $p(z) \in \mathbf{SI}_I$ if, and only if, the function $\Phi(u)$ defined in (2.3) maps the upper half-plane to the lower half-plane and has exactly l positive poles ($l = r$ in this case). This is equivalent to the inequalities (2.17)–(2.18), according to Lemmata 2.4–2.5 and formulæ (2.15)–(2.16).

If $n = 2l + 1$, then according to Theorem 2.2, $p(z) \in \mathbf{SI}_I$ if, and only if, the function $\Phi(u)$ maps the upper half-plane to the lower half-plane and has exactly r ($r = l - 1$ in this case) positive poles and one pole at zero (l poles in total). By Lemmata 2.4 and 2.6 and by formulæ (2.15)–(2.16), this is equivalent to the inequalities (2.17)–(2.18), as required. \square

Note that (2.17) is equivalent to the following inequalities

$$\Delta_{2i-1}(p) \Delta_{2i+1}(p) < 0, \quad i = 0, 1, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor,$$

where $\Delta_{-1}(p) \equiv 1$.

Following [8, Chapter 3] we note that if the function $\Phi(u)$ maps the upper half-plane to itself, then we can simplify the conditions (2.13) and (2.14) using the coefficients of the denominator of $\Phi(u)$. Indeed, if a polynomial

$$q(z) = d_0 z^m + d_1 z^{m-1} + \cdots + d_{m-1} z + d_m, \quad d_0 > 0,$$

has only real roots, then by the Descartes Rule of Signs (see e.g. [17, Part V, Chapter 1]) all its roots are positive if, and only if,

$$d_{k-1} d_k < 0, \quad k = 1, \dots, m,$$

or, that is the same,

$$(-1)^k d_k < 0, \quad k = 0, 1, \dots, m.$$

This remark together with Theorem 2.2 and Lemma 2.4 immediately implies an analogue of the stability criterion due to Liénard and Chipart [14], [6, p. 221] (see also [8] and references there).

Theorem 2.7. *The polynomial p defined in (2.1) belongs to \mathbf{SI}_I if, and only if,*

$$(-1)^j \Delta_{2j-1}(p) > 0, \quad j = 1, \dots, l,$$

$$(-1)^j a_{2j} > 0, \quad j = 1, \dots, r,$$

where l and r are defined in (2.2) and (2.8), respectively.

Analogously to Theorem 11 from [6, Chap. XV, Sec. 13] and Theorem 3.34 from [8], one can easily establish three additional similar criterions for a real polynomial to be self-interlacing. We leave these simple exercises to the reader.

We end this section with the following remark.

Remark 2.8. A polynomial $p(z)$ of degree n can always be represented as follows

$$p(z) = \widehat{p}_0(z) + \widehat{p}_1(z),$$

where

$$(2.19) \quad \widehat{p}_0(z) = \frac{p(z) + (-1)^n p(-z)}{2} = a_0 z^n + a_2 z^{n-2} + a_4 z^{n-4} + \cdots,$$

and

$$(2.20) \quad \widehat{p}_1(z) = \frac{p(z) - (-1)^n p(-z)}{2} = a_1 z^{n-1} + a_3 z^{n-3} + a_5 z^{n-5} + \cdots$$

If $\widehat{p}_0(z)$ and $\widehat{p}_1(z)$ have real interlacing zeroes, then the polynomial $p(z) = \widehat{p}_0(z) + \widehat{p}_1(z)$ has real zeroes as a linear combination of polynomials with real interlacing zeroes [2] (see also [16]). However, this notice does not help to investigate the self-interlacing property of polynomials. At the same time, this fact shows that roots of the self-interlacing polynomial $p(z)$ interlace both roots of $\widehat{p}_0(z)$ and roots of $\widehat{p}_1(z)$, while the interlacing of roots of $\widehat{p}_0(z)$ and $\widehat{p}_1(z)$, in this case, follows from the property of the function $z\Phi(z^2)$ that maps the upper half-plane to itself [8, Chapter 3].

3. INTERRELATION BETWEEN HURWITZ STABLE AND SELF-INTERLACING POLYNOMIALS

In this section we establish that the classes of self-interlacing polynomials of kind I , \mathbf{SI}_I , and *real* Hurwitz stable polynomials (the polynomials whose roots lie in the open left half-plane) are isomorphic. This actually means that the set of all stable real polynomials isomorphically embedded into the set of all real polynomials with real simple roots.

Theorem 3.1. *A real polynomial*

$$(3.1) \quad p(z) = \sum_{k=0}^n a_k z^k$$

belongs to the class \mathbf{SI}_I if, and only if, the polynomial

$$(3.2) \quad q(z) = \sum_{k=0}^n (-1)^{\frac{k(k+1)}{2}} a_k z^{n-k}$$

is Hurwitz stable.

Definition 3.2. We call the polynomial $p \in \mathbf{SI}_I$ defined in (3.1) and the Hurwitz stable polynomial q defined in (3.2) *dual* to each other.

Proof of Theorem 3.1. Indeed, by Theorem 2.2, the polynomial $p \in \mathbf{SI}_I$ if, and only if, the function (2.3)

$$\Phi(u) = \frac{a_1 u^{l-1} + a_3 u^{l-2} + a_5 u^{l-2} + \dots}{a_0 u^l + a_2 u^{l-1} + a_4 u^{l-2} + \dots}, \quad l = \left\lfloor \frac{n+1}{2} \right\rfloor,$$

maps the upper half-plane to itself and has only positive (for $n = 2r$) or nonnegative (for $n = 2r + 1$) poles. In its turn, it is equivalent to the fact that the function

$$\Psi(u) = \Phi(-u) = \frac{-a_1 u^{l-1} + a_3 u^{l-2} - a_5 u^{l-2} + \dots}{a_0 u^l - a_2 u^{l-1} + a_4 u^{l-2} + \dots} = \sum_{j=0}^{+\infty} \frac{t_j}{z^{j+1}}$$

maps the upper half-plane to the lower half-plane and has only negative (for $n = 2r$) or nonpositive (for $n = 2r + 1$) poles. Moreover,

$$z\Psi(z^2) = \frac{q(z) - (-1)^n q(-z)}{q(z) + (-1)^n q(-z)},$$

so $\Psi(u)$ is the function associated with the polynomial q . From the results of [8, Chapter 3] (see also references there) it follows that $\Psi(u)$ maps the upper half-plane to the lower one if, and only if, the Hankel determinants $D_k(\Psi)$ constructed with the coefficients t_j of the Laurent series of Ψ at infinity satisfy the inequalities

$$D_j(\Psi) > 0, \quad j = 1, \dots, l,$$

so from the formula (2.15) applied to the function $\Psi(u)$ and the polynomial \tilde{p} we obtain

$$(3.3) \quad \Delta_{2j-1}(q) > 0, \quad j = 1, \dots, l.$$

Moreover, by Corollaries 1.4 and 3.10 of [8] the function $\Psi(u)$ maps the upper half-plane to itself if, and only if, the following inequalities hold

$$(-1)^j \hat{D}_j(\Psi) > 0, \quad j = 1, \dots, r,$$

so from (2.16) we have

$$(3.4) \quad \Delta_{2j}(q) > 0, \quad j = 1, \dots, r.$$

The inequalities (3.3) and (3.4) mean that all the Hurwitz determinants $\Delta(q)$ of the polynomial q are positive, so q is Hurwitz stable by Hurwitz stability criterion [6]. \square

Remark 3.3. According to a formula in the proof of [8, Corollary 3.12], $|D_j(\Phi)| = |D_j(\Psi)|$, $j = 1, \dots, l$, and $|\hat{D}_j(\Phi)| = |\hat{D}_j(\Psi)|$, $j = 1, \dots, r$. Thus, from the formulæ (2.15)–(2.16), we have

$$|\Delta_i(p)| = |\Delta_i(q)|, \quad i = 1, \dots, n,$$

so the Hurwitz minors of the dual polynomials p and q differ only by signs. In Section 4 we prove that *all* the Hurwitz minors of the polynomials p and q possess the same property.

As an immediate consequence of Theorem 3.1, we obtain the following analogue of Stodola's theorem claiming that all the coefficients of a real stable polynomial are of the same sign [6].

Theorem 3.4. *If the polynomial p defined in (3.1) belongs to the class \mathbf{SI}_I , then*

$$(3.5) \quad (-1)^{\frac{j(j+1)}{2}} a_j > 0, \quad j = 0, 1, \dots, n.$$

Proof. In fact, if the polynomial p is self-interlacing, then by Theorem 3.1 the polynomial q defined in (3.2) is Hurwitz stable. But by Stodola's theorem its coefficients are positive (since $a_0 > 0$), that implies (3.5). \square

Remark 3.5. Theorem 3.4 was proved in [4] by another method (see [4, Lemma 2.6]).

Let us point out at one more interesting connection between Hurwitz stable and self-interlacing polynomials. To do this we need the following simple fact.

Proposition 3.6. *If polynomial $p \in \mathbf{SI}_I$, then the Hurwitz stable polynomial q defined in (3.2), can be represented as follows*

$$(3.6) \quad q(z) = i^{-n} p(iz) \frac{1-i}{2} + i^n p(-iz) \frac{1+i}{2}.$$

Proof. Let $n = 2r$. Then the polynomial $p(z)$ can be represented a sum of two polynomials

$$p(z) = p_0(z^2) + zp_1(z^2),$$

where

$$(3.7) \quad \begin{aligned} p_0(u) &= a_0 u^r + a_2 u^{r-1} + \dots + a_{2r}, \\ p_1(u) &= a_1 u^{r-1} + a_3 u^{r-2} + \dots + a_{2r-1}. \end{aligned}$$

So the corresponding dual Hurwitz stable polynomial $q(z)$ has the form

$$(3.8) \quad q(z) = (-1)^r [p_0(-z^2) + zp_1(-z^2)].$$

On the other hand, we have

$$\begin{aligned} i^{-n} p(iz) &= (-1)^r p(iz) = (-1)^r [p_0(-z^2) + izp_1(-z^2)], \\ (-i)^{-n} p(-iz) &= i^n p(-iz) = (-1)^r p(-iz) = (-1)^r [p_0(-z^2) - izp_1(-z^2)], \end{aligned}$$

that implies

$$(3.9) \quad \begin{aligned} (-1)^r p_0(-z^2) &= \frac{i^{-n} p(iz) + i^n p(-iz)}{2}, \\ (-1)^r p_1(-z^2) &= -i \frac{i^{-n} p(iz) - i^n p(-iz)}{2}. \end{aligned}$$

The formula (3.6) now follows from (3.8) and (3.9).

The case of $n = 2r+1$ can be proved analogously. The only difference we should take into account is that $p_0(u) = a_1 u^r + a_3 u^{r-1} + \dots$ and $p_1(u) = a_0 u^r + a_2 u^{r-1} + \dots$, so $q(z) = (-1)^{r+1} [p_0(-z^2) - zp_1(-z^2)]$. \square

Clearly, the converse formula is also true

$$p(z) = i^{-n} q(iz) \frac{1-i}{2} + i^n q(-iz) \frac{1+i}{2}.$$

due to the duality.

Using Theorem 3.1 and Proposition 3.6 one can establish the following curious fact.

Theorem 3.7. *Let $p \in \mathbf{SI}_l$ and let q be its dual Hurwitz stable polynomial. Then*

$$p(\lambda) = 0 \iff \arg q(i\lambda) = (-1)^{n-1} \frac{\pi}{4} \text{ or } (-1)^{n-1} \frac{5\pi}{4};$$

and, respectively, for $\mu \in \mathbb{R}$,

$$q(\mu) = 0 \iff \arg p(i\mu) = (-1)^{n-1} \frac{\pi}{4} \text{ or } (-1)^{n-1} \frac{5\pi}{4}.$$

In other word, if $p(\lambda) = 0$, then $\operatorname{Re} q(i\lambda) = \operatorname{Im} q(i\lambda)$ or $\operatorname{Re} q(i\lambda) = -\operatorname{Im} q(i\lambda)$, and if $q(\mu) = 0$, $\mu \in \mathbb{R}$, then $\operatorname{Re} p(i\mu) = \operatorname{Im} p(i\mu)$ or $\operatorname{Re} p(i\mu) = -\operatorname{Im} p(i\mu)$.

Proof. Let first the degree of p be even: $n = 2r$. Then $p(\lambda) = 0$ if, and only if, $\frac{\lambda p_1(\lambda^2)}{p_0(\lambda^2)} = -1$, where p_0 and p_1 are defined in (3.7). From (3.8) we obtain $q(i\lambda) = (-1)^r [p_0(\lambda^2) + i\lambda p_1(\lambda^2)]$. Consequently, $\arg q(i\lambda) = \arctan\left(\frac{\lambda p_1(\lambda^2)}{p_0(\lambda^2)}\right) = \arctan(-1) = -\frac{\pi}{4}$ or $\frac{3\pi}{4}$.

The case $n = 2r + 1$ can be proved analogously with the difference that $\arg q(i\lambda) = \arctan(1)$ (see the proof of Proposition 3.6). The second assertion of the theorem follows from the first one, since the polynomials $p(z)$ and $q(z)$ are dual. \square

Remark 3.8. Theorem 3.7 can be generalized. Indeed, let $p(z)$ be an *arbitrary* real polynomial as in (3.1) with $a_n \neq 0$, and let $q(z)$ is defined as in (3.2). In the same way as used in the proof of Theorem 3.7, it is easy to establish that if $\lambda \in \mathbb{R}$, then

$$p(\lambda) = 0 \iff \arg q(i\lambda) = (-1)^{n-1} \frac{\pi}{4} \text{ or } (-1)^{n-1} \frac{5\pi}{4}.$$

This fact is related to the following formula

$$(3.10) \quad \tan \sum_{k=1}^n \arctan(a_k) = \frac{\sum_{k=1}^n a_k - \sum_{i=1}^{l-1} e_{2i+1}(a_1, \dots, a_n)}{1 - \sum_{i=1}^r e_{2i}(a_1, \dots, a_n)}$$

where l and r are defined in (2.2) and in (2.8), respectively, and

$$e_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k},$$

is the k th symmetric function.

Formula (3.10) can be proved by induction from the well-known formula (see e.g. [5]):

$$\tan(\arctan(x) + \arctan(y)) = \frac{x + y}{1 - xy}.$$

If now we represent the polynomial p as follows

$$p(z) = a_0 \prod_{k=1}^n (z - \lambda_k),$$

then for $\lambda \in \mathbb{R}$ such that $\arg p(i\lambda) = (-1)^{n-1} \frac{\pi}{4}$ or $(-1)^{n-1} \frac{5\pi}{4}$, we have, by (3.10),

$$\tan(\arg p(i\lambda)) = \tan \sum_{k=1}^n \arctan\left(\frac{\operatorname{Im} \lambda_k - \lambda}{\operatorname{Re} \lambda_k}\right) = \frac{\lambda p_1(-\lambda^2)}{p_0(-\lambda^2)} = (-1)^{n-1},$$

Now from the formula $q(z) = (-1)^l [p_0(-z^2) + (-1)^n z p_1(-z^2)]$, it follows that $q(\lambda) = 0$.

Also, formula (3.10) can be used to find roots of the dual polynomial q if the roots of p are known.

Example 3.9. Consider the polynomial

$$p(z) = (z + a)^n = \sum_{k=0}^n \binom{n}{k} z^k a^{n-k}$$

which is stable for $a > 0$. By Theorem 3.7, its dual polynomial

$$q(z) = \sum_{k=0}^n (-1)^{\frac{k(k+1)}{2}} \binom{n}{k} z^k a^{n-k}$$

has n distinct *real* roots μ_k , $k = 1, \dots, n$, satisfying the condition

$$\arg p(i\mu_k) = n \arctan \left(\frac{\mu_k}{a} \right) = (-1)^{n-1} \left(\frac{\pi}{4} + \pi k \right).$$

Therefore,

$$\mu_k = (-1)^{n-1} a \cdot \tan \frac{\pi(4k+1)}{4n}, \quad k = 1, \dots, n.$$

Note that if a is an arbitrary non-zero complex number, the roots of the polynomial q have the same form.

In a similar way, it is possible to find the roots of the self-interlacing polynomial $q(z)$ dual to a stable polynomial with only n distinct roots of equal multiplicity. We leave such an exercise to the reader.

4. PROPERTIES OF SELF-INTERLACING POLYNOMIALS

Hurwitz stable polynomials have interpretation in terms of Stieltjes continued fractions [6, Chapter XV, §14, Theorem 16]. Due to the relation between Hurwitz stable polynomials and self-interlacing polynomials provided by Theorem 3.1, it is possible to associate with self-interlacing polynomials certain continued fractions of Stieltjes type considered e.g. in [8, Section 3.4].

The following theorem presents a relation between self-interlacing polynomials and continued fractions of Stieltjes type.

Theorem 4.1. *The polynomial p of degree n belongs to the class \mathbf{SI}_I if, and only if, its associated function $\Phi(u)$ defined in (2.3) has the following Stieltjes continued fraction expansion:*

$$(4.1) \quad \Phi(u) = \frac{1}{c_1 u + \frac{1}{c_2 + \frac{1}{c_3 u + \frac{1}{\ddots + \frac{1}{c_{2r-1} u + \frac{1}{c_{2r} + \frac{1}{c_{2r+1} u}}}}}}}, \quad \text{with } (-1)^i c_i > 0, \quad i = 1, \dots, 2r,$$

where $c_{2r+1} = \infty$ if n is even, and $c_{2r+1} < 0$ if n is odd. The number r is as in (2.8).

Proof. In fact, by Theorem 2.2, $p \in \mathbf{SI}_I$ if, and only if, the function $-\Phi(u)$ maps the upper half-plane to the lower half-plane. Now the assertion of the theorem follows from Theorem 3.8 and Corollaries 3.39 and 3.40 of the work [8]. \square

According to [8], the coefficients c_i can be found by the following formulæ

$$(4.2) \quad c_i = \frac{\Delta_{i-1}^2(p)}{\Delta_{i-2}(p)\Delta_i(p)}, \quad i = 1, \dots, n.$$

This formulæ follow from (2.15)–(2.16) and from formulæ (1.113)–(1.114) of the work [8]. The signs of c_i in Theorem 4.1 follow from (4.2).

Using Theorem 2.2 one can easily obtain the following fact.

Theorem 4.2. *Let $p \in \mathbf{SI}_I$ and $\deg p \geq 2$. Then the polynomial*

$$p_j(z) = \sum_{i=0}^{n-2j} \left[\frac{n-i}{2} \right] \left(\left[\frac{n-i}{2} \right] - 1 \right) \cdots \left(\left[\frac{n-i}{2} \right] + j - 1 \right) a_i z^{n-2j-i}, \quad j = 1, \dots, r-1.$$

belongs to the class \mathbf{SI}_I . Here r is defined in (2.8).

Proof. Let $n = 2r$. By Theorems 2.2, if $p \in \mathbf{SI}_I$, then the associated function $-\Phi(u) = -\frac{p_1(u)}{p_0(u)}$, where $q_0(u)$ and $q_1(u)$ are defined in (3.7), maps the upper half-plane to the lower half-plane, and has only positive poles. This is equivalent to the fact (see [8, Theorem 3.4]) that $q_0(u)$ and $q_1(u)$ have simple, negative, and interlacing roots, and $-\Phi(u)$ is decreasing between its poles.

By V.A. Markov theorem [2, Chapter 1, Theorem 9] (see also [8, Theorem 3.8]) if two polynomials have real, simple, and interlacing roots, then their derivatives also have real, simple, and interlacing roots. Thus, for every $j = 1, \dots, r-1$, the roots of j^{th} derivatives of the polynomials $q_0(u)$ and $q_1(u)$ also have simple, positive, and interlacing roots. Moreover, since the leading coefficients of $q_0^{(j)}(u)$ and $q_1^{(j)}(u)$ are always of different signs, so the functions $-\frac{q_1^{(j)}(u)}{q_0^{(j)}(u)}$ are negative for sufficiently large positive u . Therefore, by

Theorem 2.2 the polynomials $p_j(z) = q_0^{(j)}(z^2) + zq_1^{(j)}(z^2)$ belong to the class \mathbf{SI}_I for all $j = 1, \dots, r-1$.

The case $n = 2r + 1$ can be proved analogously. \square

Using V.A. Markov's theorem it is also easy to prove the following fact.

Theorem 4.3. *If $p(z) \in \mathbf{SI}_I$, then $p^{(k)}(z) \in \mathbf{SI}_I$, $k = 1, \dots, n-1$, where $p^{(k)}(z)$ is the k^{th} derivative of $p(z)$.*

Proof. By Definition of self-interlacing polynomials $p(z)$ and $p(-z)$ have real, simple, and interlacing roots. By V.A. Markov's theorem for any $k = 1, \dots, n-1$, the polynomials $p^{(k)}(z)$ and $p^{(k)}(-z)$ also have real, simple and interlacing roots. Moreover, the largest root of $p^{(k)}(z)$ is greater than the largest root of $p^{(k)}(-z)$ (see [8, Corollary 3.7] for detailed proof of this fact), so $p^{(k)}(z) \in \mathbf{SI}_I$ for all $k = 1, \dots, n-1$. \square

Now we are in a position to study minors of the Hurwitz matrix of self interlacing polynomials. Recall that given a polynomial $p(z)$ as in (2.1) its Hurwitz matrix has the form

$$(4.3) \quad \mathcal{H}_n(p) = \begin{pmatrix} a_1 & a_3 & a_5 & a_7 & \dots & 0 & 0 \\ a_0 & a_2 & a_4 & a_6 & \dots & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & \dots & 0 & 0 \\ 0 & a_0 & a_2 & a_4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & a_{n-2} & a_n \end{pmatrix},$$

The Hurwitz minors defined in (1.6) are the leading principal minors of $\mathcal{H}_n(p)$.

Let $p \in \mathbf{SI}_I$, and let q be its associated Hurwitz stable polynomial. From the proof of Proposition 3.6 it follows that if $p(z) = p_0(z^2) + zp_1(z^2)$, where $p_0(u)$ and p_1 are the odd and even parts of p , then $q(z) = (-1)^l [p_0(-z^2) + (-1)^n zp_1(-z^2)]$, where l is defined in (2.2). The Hurwitz matrix of the polynomial q has the form

$$\mathcal{H}_n(q) = \begin{pmatrix} -a_1 & a_3 & -a_5 & a_7 & \dots & 0 \\ a_0 & -a_2 & a_4 & -a_6 & \dots & 0 \\ 0 & -a_1 & a_3 & -a_5 & \dots & 0 \\ 0 & a_0 & -a_2 & a_4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & (-1)^{\frac{n(n+1)}{2}} a_n \end{pmatrix}$$

It is easy to see that the matrix $\mathcal{H}_n(q)$ can be factorized as follows

$$(4.4) \quad \mathcal{H}_n(q) = C_n \mathcal{H}_n(p) E_n,$$

where the $n \times n$ matrices C_n and E_n have the forms

$$C_n = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad E_n = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & -1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

All the non-principal minors of these matrices equal zero. The principal minors of these matrices can be easily calculated:

$$(4.5) \quad C_n \begin{pmatrix} i_1 & i_2 & \dots & i_m \\ i_1 & i_2 & \dots & i_m \end{pmatrix} = (-1)^{\sum_{k=1}^m \frac{i_k(i_k-1)}{2}}, \quad E_n \begin{pmatrix} i_1 & i_2 & \dots & i_m \\ i_1 & i_2 & \dots & i_m \end{pmatrix} = (-1)^{\sum_{k=1}^m i_k}.$$

where $1 \leq i_1 < i_2 < \dots < i_m \leq n$. Thus, the Cauchy–Binet formula together with (4.4) and (4.5) implies

$$(4.6) \quad \mathcal{H}_n(q) \begin{pmatrix} i_1 & i_2 & \dots & i_m \\ j_1 & j_2 & \dots & j_m \end{pmatrix} = (-1)^{\sum_{k=1}^m \frac{i_k(i_k-1)}{2} + \sum_{k=1}^m j_k} \mathcal{H}_n(p) \begin{pmatrix} i_1 & i_2 & \dots & i_m \\ j_1 & j_2 & \dots & j_m \end{pmatrix},$$

where $1 \leq \begin{matrix} i_1 < i_2 < \dots < i_m \\ j_1 < j_2 < \dots < j_m \end{matrix} \leq n$.

Thus, we established that the absolute values of the corresponding minors of the Hurwitz matrices of the dual polynomials p and q are equal as we announced in Section 3.

Remark 4.4. It is clear that the formulæ (4.6) is true for two arbitrary complex polynomials p and q related as in (3.1)–(3.2).

The polynomial q is stable by assumption. Consequently, by Asner’s theorem [1] (see also [7, 8, 3, 11]), the matrix $\mathcal{H}_n(q)$ is totally nonnegative, that is, any its minor is nonnegative. From this fact and the formula (4.6) we obtain the following theorem.

Theorem 4.5. *Let a polynomial p be defined in (3.1), and let $\mathcal{H}_n(p)$ be its Hurwitz matrix defined in (4.3). If $p \in \mathbf{SI}_I$, then*

$$(-1)^{\sum_{k=1}^m \frac{i_k(i_k-1)}{2} + \sum_{k=1}^m j_k} \mathcal{H}_n(p) \begin{pmatrix} i_1 & i_2 & \dots & i_m \\ j_1 & j_2 & \dots & j_m \end{pmatrix} \geq 0,$$

where $1 \leq \begin{matrix} i_1 < i_2 < \dots < i_m \\ j_1 < j_2 < \dots < j_m \end{matrix} \leq n$. Moreover, the absolute values of the corresponding minors of the matrices $\mathcal{H}_n(p)$ and $\mathcal{H}_n(q)$, where q is the polynomials dual to p as in (3.2), are equal.

5. THE SECOND PROOF OF THE HURWITZ SELF-INTERLACING CRITERION. STABILITY CRITERION.

In this Section, we provide another approach to proof of Theorem 1.2 based in Hankel minors related to the Laurent series at infinity of the function $R(z)$ defined in (2.4) instead of the more standard function $\Phi(u)$ defined in (2.3).

Let again

$$(5.1) \quad p(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n, \quad a_1, \dots, a_n \in \mathbb{R}, \quad a_0 > 0,$$

be a real polynomial. Consider the function $R(z)$ defined in (2.4):

$$(5.2) \quad R(z) = \frac{(-1)^n p(-z)}{p(z)},$$

and expand them into their Laurent series at ∞ :

$$R(z) = 1 + \frac{s_0}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \dots,$$

According to Kronecker's theorem (see [8, p. 426] and references there), rank of the matrix $S = \|s_{j+k}\|_0^\infty$ is equal to the number of poles of the functions R . It is clear that rank of S equals n if the polynomials $p(z)$ and $p(-z)$ have no common zeroes.

Lemma 5.1. *For the functions R (5.2), the following formulæ hold:*

$$(5.3) \quad a_0^{2j} D_j(R) = (-1)^{\frac{j(j+1)}{2}} 2^j a_0 \Delta_{j-1}(p) \Delta_j(p), \quad j = 1, 2, \dots$$

where $\Delta_j(p)$ are defined in (1.6), $\Delta_0(p) \equiv 1$.

Proof. By the Hurwitz formula (see [6, p. 214], [8, Theorem 1.5] and references there) applied to the function $R(z)$, we have

$$a_0^{2j} D_j(R) = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \dots & a_{j-1} & a_j & \dots & a_{2j-2} & a_{2j-1} \\ a_0 & -a_1 & a_2 & -a_3 & \dots & -a_{j-1} & a_j & \dots & a_{2j-2} & -a_{2j-1} \\ 0 & a_0 & a_1 & a_2 & \dots & a_{j-2} & a_{j-1} & \dots & a_{2j-3} & a_{2j-2} \\ 0 & a_0 & -a_1 & a_2 & \dots & a_{j-2} & -a_{j-1} & \dots & -a_{2j-3} & a_{2j-2} \\ 0 & 0 & a_0 & a_1 & \dots & a_{j-3} & a_{j-2} & \dots & a_{2j-4} & a_{2j-3} \\ 0 & 0 & a_0 & -a_1 & \dots & -a_{j-3} & a_{j-2} & \dots & a_{2j-4} & -a_{2j-3} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_0 & a_1 & \dots & a_{j-1} & a_j \\ 0 & 0 & 0 & 0 & \dots & a_0 & -a_1 & \dots & -a_{j-1} & a_j \end{vmatrix} =$$

$$= 2^j \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \dots & a_{j-1} & a_j & \dots & a_{2j-2} & a_{2j-1} \\ a_0 & 0 & a_2 & 0 & \dots & 0 & a_j & \dots & a_{2j-2} & 0 \\ 0 & a_0 & a_1 & a_2 & \dots & a_{j-2} & a_{j-1} & \dots & a_{2j-3} & a_{2j-2} \\ 0 & a_0 & 0 & 0 & \dots & a_{j-2} & 0 & \dots & 0 & a_{2j-2} \\ 0 & 0 & a_0 & a_1 & \dots & a_{j-3} & a_{j-2} & \dots & a_{2j-4} & a_{2j-3} \\ 0 & 0 & a_0 & 0 & \dots & 0 & a_{j-2} & \dots & a_{2j-4} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_0 & a_1 & \dots & a_{j-1} & a_j \\ 0 & 0 & 0 & 0 & \dots & a_0 & 0 & \dots & 0 & a_j \end{vmatrix} =$$

$$= (-2)^j \begin{vmatrix} a_0 & 0 & a_2 & 0 & \dots & 0 & a_j & \dots & a_{2j-2} & 0 \\ 0 & a_1 & 0 & a_3 & \dots & a_{j-1} & 0 & \dots & 0 & a_{2j-1} \\ 0 & a_0 & 0 & a_2 & \dots & a_{j-2} & 0 & \dots & 0 & a_{2j-2} \\ 0 & 0 & a_1 & 0 & \dots & 0 & a_{j-1} & \dots & a_{2j-3} & 0 \\ 0 & 0 & a_0 & 0 & \dots & 0 & a_{j-2} & \dots & a_{2j-4} & 0 \\ 0 & 0 & 0 & a_1 & \dots & a_{j-3} & 0 & \dots & 0 & a_{2j-3} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_0 & 0 & \dots & 0 & a_j \\ 0 & 0 & 0 & 0 & \dots & 0 & a_1 & \dots & a_{j-1} & 0 \end{vmatrix} =$$

$$\begin{aligned}
& = (-2)^j a_0 \begin{vmatrix} a_1 & 0 & a_3 & 0 & \dots & a_{j-1} & 0 & \dots & 0 & a_{2j-1} \\ a_0 & 0 & a_2 & 0 & \dots & a_{j-2} & 0 & \dots & 0 & a_{2j-2} \\ 0 & 0 & a_1 & 0 & \dots & a_{j-3} & 0 & \dots & 0 & a_{2j-3} \\ 0 & 0 & a_0 & 0 & \dots & a_{j-4} & 0 & \dots & 0 & a_{2j-4} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_1 & 0 & \dots & 0 & a_{j+1} \\ 0 & 0 & 0 & 0 & \dots & a_0 & 0 & \dots & 0 & a_j \\ 0 & a_1 & 0 & a_3 & \dots & 0 & a_{j-1} & \dots & a_{2j-3} & 0 \\ 0 & a_0 & 0 & a_2 & \dots & 0 & a_{j-2} & \dots & a_{2j-4} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & a_2 & \dots & a_j & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & a_1 & \dots & a_{j-1} & 0 \end{vmatrix} = \\
& = (-2)^j a_0 (-1)^{\frac{j(j-1)}{2}} \begin{vmatrix} a_1 & a_3 & a_5 & \dots & a_{2j-1} & 0 & 0 & \dots & 0 & 0 \\ a_0 & a_2 & a_4 & \dots & a_{2j-2} & 0 & 0 & \dots & 0 & 0 \\ 0 & a_1 & a_3 & \dots & a_{2j-3} & 0 & 0 & \dots & 0 & 0 \\ 0 & a_0 & a_2 & \dots & a_{2j-4} & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{j+1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & a_j & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & a_1 & a_3 & \dots & a_{2j-5} & a_{2j-3} \\ 0 & 0 & 0 & \dots & 0 & a_0 & a_2 & \dots & a_{2j-6} & a_{2j-4} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & a_{j-2} & a_j \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & a_{j-3} & a_{j-1} \end{vmatrix} = \\
& = (-1)^{\frac{j(j+1)}{2}} 2^j a_0 \Delta_{j-1}(p) \Delta_j(p).
\end{aligned}$$

Here we set $a_j = 0$ for $j > n$. □

Remark 5.2. Note that the formulæ (5.3) can be obtained (overcoming certain difficulties) from some theorems of the book [21]. But it is simpler to deduce them directly as we have done above.

Using this lemma it is easy to prove the equivalence of the conditions 1) and 2) of Theorem 1.2.

The second proof of Theorem 1.2. According to Theorem 2.1, $p \in \mathbf{SI}_I$ if, and only if, the function $R(z)$ maps the upper half-plane of the complex plane to the lower half-plane, and has exactly n poles. Consequently (see e.g. [8, Theorem 3.4] and references there), the following inequalities hold

$$D_j(R) > 0, \quad j = 1, \dots, n,$$

so from (5.3) we obtain

$$(5.4) \quad (-1)^{\frac{j(j+1)}{2}} \Delta_{j-1}(p) \Delta_j(p) > 0, \quad j = 1, \dots, n.$$

Multiplying the inequalities (5.4) for $j = 2m$ and $j = 2m - 1$, we obtain

$$\Delta_{2m-1}^2 \Delta_{2m} \Delta_{2m-2} > 0.$$

Consequently, the minors $\Delta_{2i}(p)$, $i = 1, \dots, r$, are positive, since $\Delta_0(p) = 1$, so the inequalities (2.18) hold.

If we multiply the inequalities (5.4) for $j = 2m$ and $j = 2m + 1$, we get

$$-\Delta_{2m}^2(p) \Delta_{2m-1}(p) \Delta_{2m+1}(p) > 0.$$

These inequalities imply (2.17).

The converse assertion can be proved in the same way. That is, the inequalities (2.17)–(2.18) imply the inequalities (5.4) which, in turn, imply the inequalities $D_j(R) > 0$, $j = 1, \dots, n$, according to (5.3). According to [8, Theorem 3.4], the function $R(z)$ maps the upper half-plane of the complex plane to the lower half-plane, and has exactly n poles, so $p \in \mathbf{SI}_I$ by Theorem 2.1, as required. □

From Lemma 5.1 and from Hurwitz's stability criterion claiming the positivity of the Hurwitz minors $\Delta_j(p)$, $j = 1, \dots, n$, for any Hurwitz stable polynomial of degree n and vice versa ([10, 6], see also [12] and references there) we get the following stability criterion.

Theorem 5.3. *For a real polynomial p of degree n , the following statements are equivalent:*

- (1) *The polynomial p is Hurwitz stable.*
- (2) *The function $R(z)$ defined in (5.2) maps the open right half-plane of the complex into the open unit disc and has exactly n poles.*
- (3) *For the function R , the following inequalities hold*

$$(-1)^{\frac{j(j+1)}{2}} D_j(R) > 0, \quad j = 1, 2, \dots, n.$$

Proof. The equivalence (1) and (3) follows from the Hurwitz stability criterion and Lemma 5.1 as we mentioned above.

Let $p(z)$ is Hurwitz stable. Then $p(z)$ and $p(-z)$ have no common roots, so $R(z)$ has exactly n poles. The polynomial $p(z)$ can be represented as follows

$$p(z) = a_0 \prod_k (z - \lambda_k) \prod_j (z - \mu_j)(z - \bar{\mu}_j),$$

where $\lambda_k < 0$, $\operatorname{Re} \mu_j < 0$.

Consequently, the function $R(z)$ has the form

$$R(z) = \prod_k \frac{z + \lambda_k}{z - \lambda_k} \cdot \prod_j \frac{z + \mu_j}{z - \bar{\mu}_j} \cdot \frac{z + \bar{\mu}_j}{z - \mu_j}.$$

It is clear now that for any z such that $\operatorname{Re} z > 0$ we have $|R(z)| < 1$, since

$$\left| \frac{z + a}{z - \bar{a}} \right| < 1,$$

for any z and a such that $\operatorname{Re} z > 0$ and $\operatorname{Re} a < 0$, and $R(z)$ is a product of such functions. Additionally, it is easy to see that $R(z)$ maps the imaginary axis into the unit circle and the open left half-plane into the exterior of the closed unit disc.

Conversely, if $R(z)$ maps the open right half-plane into the unit circle, and has exactly n poles, then it has no poles in the open right half-plane. It also has no poles on the imaginary axis, since any pure imaginary zero of $p(z)$ is a zero of $p(-z)$, but they have no common zeroes by assumption. So all poles of $R(z)$ (the zeroes of $p(z)$) lie in the open left half-plane, as required. \square

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